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FURTHER DEVELOPMENT OF THE THEORY OF MULTICOMPONENT DRY FRICTION

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Abstract. *It is proposed further development of the theory of multicomponent dry friction which consists in presenting a more convenient form for the problems of the dynamics of the earlier developed models with a reduced number of coefficients.*

1 INTRODUCTION

In recent years there has been a surge of interest in the dynamics of systems of solids with friction under conditions of the combined kinematics. Fact is that researchers in the field of dry friction has long been known that in case of combined of the kinematics, when the rubbed solids are participated, simultaneously, in the sliding, spinning and rolling motion, the use of the classical Coulomb's law is not correct, and the friction law is undergoing significant changes.

One of the first attempts to describe the relationship of friction and spinning in the case of non-point contact of moving solids was undertaken by Contensou Contensou got numerical dependence of the dry friction force from the ratio of the slip velocity to the linear velocity of spinning.

The principle new development of the theory was given by Zhuravlev in With the aid of transition of the coordinaton system origin, to the instantaneous center of velocities, he obtained exact analytical expressions of the resultant vector and friction torque for circular contact spot, on the assumption that the distribution of contact pressure in the contact area subject to the law of Hertz. To use the obtained relationships in the dynamics problems, Zhuravlev built their fractional-linear but Pade. The convenience of the use of the Pade approximations making it possible to describe the effects of the combined dry friction for the entire range of angular and linear velocities has led in subsequent to the development of principally new models of friction on their basis [3-10].

Shortly after the publication of [2] in Russia and some European countries, several research groups focus their attention on studying the effects of dry friction in the combined kinematics. Main subjects of their research were the construction of mathematical models of dry friction, explaining the conditions of equilibrium of solids with dry friction and solution of various problems of classical dynamics. The narrow field of theoretical mechanics that began in the early 2000's with a few publications by Zhuravlev [2], Ivanov [11], Kireenkov [12], Leine [13] and Karapetyan [14] are formed in integral scientific direction. Dozens publications on the subject were published.

Starting with the work [12] author of this publication began to develop a theory of the multi-component dry friction, one of the directions of it consist in the construction of the phenomenological dry friction models which are suitable for using in differential equations of motion.

The main distinguishing feature of this approach is that, at first, under the assumption of validity both the classical Coulomb's law in differential form for small element of the area inside the contact area and its generalized forms, there are constructed the exact coupled integral dry friction model, obtained by integrating the differentials of the principal vector and torque on the contact spot.

It is worth explain the used of the term "Exact integral model" because any model can not be exact, because it is only an approximation to a real phenomenon. This notion is used in the sense that, after the initial assumptions about the validity of Coulomb's law in classical and generalized differential form and general properties of the normal contact distributions inside of contact spot, all other computations, from a mathematical point of view, are being made exactly, without the use of approximate methods. Thus, after writing expressions for the differentials of the dry friction principal vector and torque, all subsequent transformations are exact results, reflecting the nature of the phenomenon.

The integral model gives a good description of the dry friction effects in the case of combined kinematics, but is inconvenient to be used in problems of dynamics, because it is required to calculate multiple integrals in the right-hand sides of the equations of motion. Previously, to escape this difficult procedure, the exact integral models are replaced by approximate models based on the Pade expansions of the first or second order. These replacements substantially simplify the combined dry friction modeling, making the calculation of double integrals over the contact area unnecessary. But in the case of arbitrary (in sign) velocities spinning ω and sliding v, the approximated models contain non-smooth functions (modules of velocities ω and v) in the denominators of the corresponded Pade expansions. A new type of approximated models which are the ratio of the linear form to square root of the quadratic form makes it possible to avoid this inconvenience.

2 EXACT INTEGRAL MODELS IN THE CASE OF COMBINED KINEMATICS

The dry friction exact integral models in the case of simultaneously sliding, spinning and rolling are constructed for circular contact sites under the assumption that the Coulomb law in generalized differential form holds for the small surface element dS in the interior of the contact spot, according to which the differentials of the resultant vector $d\mathbf{F}$ and the moment of friction dM_C with respect to the contact spot center are determined by the formulas [8]:

$$d\mathbf{F} = -f\sigma\frac{\mathbf{V}}{|\mathbf{V}|}\left(1+\mu_1|\mathbf{V}|^3-\mu_2|\mathbf{V}|\right)dS,\, dM_C = -f\sigma\frac{\mathbf{r}\times\mathbf{V}}{|\mathbf{V}|}\left(1+\mu_1|\mathbf{V}|^3-\mu_2|\mathbf{V}|\right)dS, \quad (1)$$
$$\mathbf{V} = (v-\omega y, \omega x),\, \mathbf{r} = (x, y)$$

where f is the coefficient of friction, $\mathbf{r} = (x, y)$ is the position vector of an elemental area in the interior of the contact spot with respect to its center Fig. 1 (left), ω is the angular velocity of rotation of the contact spot center, but μ_1 and μ_2 are the coefficients which can be defined in practice from experiments.. Necessity of using of the generalized differential form of the Coulomb law is caused by the numerous experimental investigations [15].

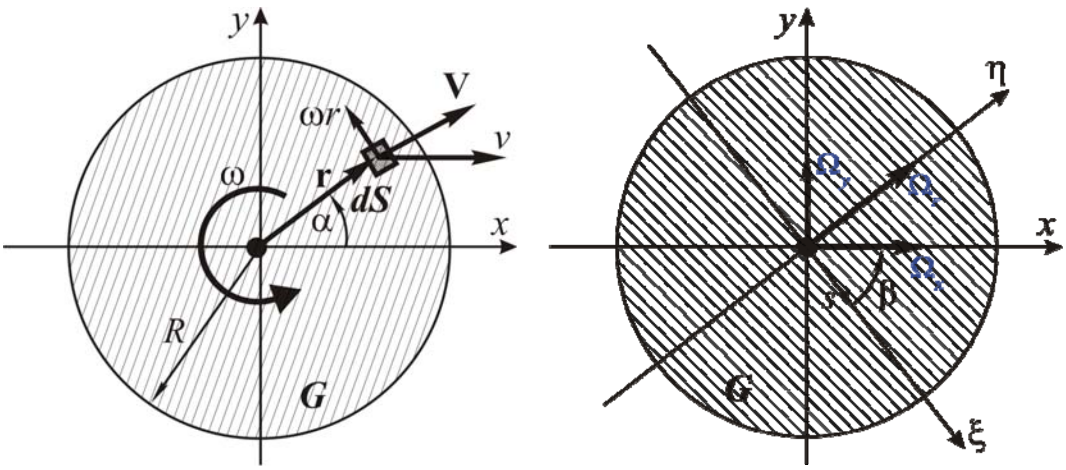

Figure 1. Kinematics inside the contact spot

In addition, in process of the exact integral models construction there are is used well known results from the theory of elasticity that tangent stresses lead to shift in the symmetric diagram of the normal contact stresses in the direction of the instantaneous sliding velocity v or in the rolling direction.

To use these results in the dynamics problems, it is proposed the simple asymptotic representations for the contact stresses distributions based on their general properties [6-7,16]:

$$\sigma(x, y) = \sigma_0 \left(1 + k_x x/R + k_y y/R\right) \tag{2}$$

where R - radius of contact spot and where $\sigma_0 = \sigma_0(r)$ - distribution of normal contact stresses at absence of motion having the properties of central symmetry

Presence of the simultaneously sliding and rolling in the different directions leads only to summarization of the corresponded coefficients. The influence of each of these effects can be investigated, due to linearity, independently. The only difference is that in case of rolling these coefficients depend on the direction and module of the rolling velocity Ω_r Fig 1 (right):

$$k_{xr} = \frac{k_r \Omega_y}{R\Omega_r},\, k_{xr} = -\frac{k_r \Omega_x}{R\Omega_r},\, \Omega_r = \sqrt{\Omega_x^2 + \Omega_y^2},\ \left|k_r\right| \le 1,\, k_r \equiv 0 \text{ if } \Omega_r = 0 \tag{3}$$

To define the corresponded coefficients can be used or the results of theory of elasticity [6] or procedure developed in [6].

Thus, we have substantial approximation to the real situation in dependence on the general properties of the normal contact stresses distribution and real differential characteristics of the friction law.

Integration of the corresponded differentials over the contact spot yields the resultant vector $\mathbf{F}$ of the friction force and torque $\mathbf{M_C}$:

$$
\begin{aligned}
&\mathbf{F} = -f\iint_G \sigma(x,y)\frac{\mathbf{V}}{|\mathbf{V}|}dxdy - f\iint_G \sigma(x,y)\mathbf{V}\left(\mu_1\mathbf{V}^2-\mu_2\right)dxdy \\
&\mathbf{M_C} = -f\iint_G \sigma(x,y)\frac{\mathbf{r}\times\mathbf{V}}{|\mathbf{V}|}dxdy - f\iint_G \sigma(x,y)\mathbf{r}\times\mathbf{V}\left(\mu_1\mathbf{V}^2-\mu_2\right)dxdy \\
&G=\{(x,y): x^2+y^2\le R^2\},\ \mathbf{F}=(F_{\|},F_{\perp})
\end{aligned}
\tag{4}
$$

where $F_{\|}$ and $F_{\perp}$ denote the respective components of the resultant vector directed along the tangent and the normal to the trajectory of motion modulus of which has form:

$$
\begin{aligned}
&F_{\|} = f\iint_G\left(\frac{(v-\omega y)(1+k_y\, y/R)}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega v y}}+\mu_1 v^3-\mu_2 v+2\mu_1 v\omega^2(x^2+y^2)\right)\sigma_0 dxdy \\
&F_{\perp} = \frac{k_x f}{R}\iint_G \frac{\omega x^2\sigma_0}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega v y}}dxdy,\ G=\{(x,y): x^2+y^2\le R^2\} \\
&M_C = f\iint_G \frac{(\omega(x^2+y^2)-vy)(1+k_y\, y/R)}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega v y}}\sigma_0 dxdy
\end{aligned}
\tag{5}
$$

Expressions (5) are calculated under supposition that influence of nonlinearity in the Coulomb law on the friction torque is neglegible and the coefficients k_x, k_y are the small parameters. These dependencies present the exact dry friction integral model in the case of combined kinematics.

It is convinient to present the double integrals in formulas (5) in the polar coordinates: $x=r\cos\alpha,\ y=r\sin\alpha,\ r\in[0,1],\ \alpha\in[0,2\pi]$ (Fig. 1) with origin in the contact spot center. In these coordinates, the polynomials terms in expressions for the friction force component $F_{\|}$ directed along the tangent to the trajectory of motion are the first moments of the normal contact stresses distribution of the first and third orders [7, 10].

3 APPROXIMATED DRY FRICTION MODELS

The exact integral model (5) gives a good description of the dry friction effects in the case of combined kinematics, but is inconvenient to be used in problems of dynamics, because it is required to calculate multiple integrals in the right-hand sides of the equations of motion. Previously, to escape this difficult procedure, the exact integral models are replaced by approximate models based on the Pade expansions of the first or second order. These replacements substantially simplify the combined dry friction modeling, making the calculation of double integrals over the contact area unnecessary. But in the case of arbitrary (in sign) velocities ω and v, the approximated models contain non-smooth functions (modules of velocities ω and v) in the denominators of the corresponded Pade expansions. A new type of approximated models which are the ratio of the linear form to square root of the quadratic [16] form makes it possible to avoid this inconvenience. Possibility to use this kind of expansions was first mentioned in [9].

Procedures of the approximated model construction are based on the analytical properties of the double integrals (first terms in proposed integral model) as functions of the velocities $u = \omega R$ and v.

$$
\begin{aligned}
F_{\|}(u,v) &= fN\int_0^{2\pi}\int_0^1 \frac{(v - ur\sin\alpha)r(1+k_y r\sin\alpha)\sigma_0(r)}{\sqrt{u^2r^2+v^2-2uvr\sin\alpha}}\,drd\alpha \\
F_{\perp}(u,v) &= k_x fN\int_0^{2\pi}\int_0^1 \frac{ur^3\sigma_0(r)\cos^2\alpha}{\sqrt{u^2r^2+v^2-2uvr\sin\alpha}}\,drd\alpha \\
M_C(u,v) &= fRN\int_0^{2\pi}\int_0^1 \frac{(ur^2 - vr\sin\alpha)r(1+k_y r\sin\alpha)\sigma_0(r)}{\sqrt{u^2r^2+v^2-2uvr\sin\alpha}}\,drd\alpha
\end{aligned}
\tag{6}
$$

These integrals can be considered independently due to additivity of the integral model (5) with the aid of formal zeroing of the parameters μ_1 and μ_2 - case correspondents to the classical Coulomb low in differential form. Using Coulomb's law in generalized differential form leads only to appearances of the additional polynomial terms.

One of the main analytical properties the exact integral expressions (6) is that the absolute values of the friction force components and toque are the homogeneous functions of the variables u and v of zero order of homogeneity: $F_{\|}(\lambda u,\lambda v) = \lambda^0 F_{\|}(\lambda u,\lambda v)$, $F_{\perp}(\lambda u,\lambda v) = \lambda^0 F_{\perp}(\lambda u,\lambda v)$, $M_C(\lambda u,\lambda v) = \lambda^0 M_C(\lambda u,\lambda v)$ Consequently, their approximations have to be the homogeneous functions of the variables u and v of zero order of homogeneity. This fact significantly reduces the possible type of approximations, coefficients of which are defined from the behavior $F_{\|}$, $F_{\perp}$ and M_C as functions variables $\{u,v\}$ as well as the behavior of their first derivatives at zero and at infinity . In result, the approximated analytical dry friction model in case of using of the Coulomb law classical differential form is

$$
M_C = \frac{M_0(u+k_y m_1 v)}{\sqrt{u^2+mv^2}},\quad F_{\|} = \frac{F_0(v+k_y a_1 u)}{\sqrt{v^2+au^2}},\quad F_{\perp} = \frac{k_x b_1 u}{\sqrt{(u^2+bv^2)}}
\tag{7}
$$

and in case of using of the Coulomb law in generalized differential form is

$$
\begin{aligned}
F_{\|} &= \frac{F_0(v+k_y a_1 u)}{\sqrt{v^2+au^2}} + 2\pi F_0\left((\mu_1 v^3 - \mu_2 v)I_1 + 2\mu_1 vu^2 I_3\right),\ F_{\perp} = \frac{k_x b_1 u}{\sqrt{(u^2+bv^2)}} \\
M_C &= \frac{M_0(u+k_y m_1 v)}{\sqrt{u^2+mv^2}},\quad I_1 = \int_0^1 r\sigma_0(r)dr,\ I_3 = \int_0^1 r^3\sigma_0(r)dr
\end{aligned}
\tag{8}
$$

Coefficients of these models (7-8) can be calculated analytically [6-10] or numerically if the distribution of the normal contact stresses inside of contact spot is described by a priory known laws:

$$\frac{1}{m}=\left(\frac{v}{M_0}\frac{\partial M_C}{\partial u}\bigg|_{u=0}\right)^2, \frac{1}{a}=\left(\frac{u}{F_0}\frac{\partial F_{\|}}{\partial v}\bigg|_{v=0}\right), \frac{1}{b}=\left(\frac{v}{\mu F_0}\frac{\partial F_{\perp}}{\partial u}\bigg|_{u=0}\right)^2 \tag{9}$$

where functions $F_{\|}$, $F_{\perp}$ and M_C are defined by the formulas (6). In the other cases they can be estimated from the experiments [15].

The dynamics coupling of the dry friction models (5), (7-8) is defined by the coefficient k_x. If the external forces are absent then it can be calculated from the simple equation [10]: $F_{\|}h = Ns$, where h - distance from the center mass of the moving solids to the plane of sliding and s - the shifting of the center of gravity of the contact spot in the direction of sliding or rolling (Fig. 1). In result, the approximated analytical dry friction model has form:

$$\begin{gathered} F_{\|} = F_0\left(\frac{v+k_y a_1 u}{\sqrt{v^2+au^2}} + 2\pi\left((\mu_1 v^3 - \mu_2 v)I_1 + 2\mu_1 v u^2 I_3\right)\right) \\ F_{\perp} = \frac{\mu_x F_0 uv}{\sqrt{(u^2+bv^2)(v^2+au^2)}}, \; M_C = M_0\frac{u+k_y m_1 v}{\sqrt{u^2+mv^2}} \end{gathered} \tag{10}$$

New kind of the approximated dry friction models permits to escape using not smooth functions in the cases when velocities are changed their signs. These models are are defined by the same coefficients amounts as models based on Pade expansions and completely satisfies to the all integral model analytical properties. Moreover, the accuracy of these models are corresponded to accuracy of the models based on the second order Pade expansions

4 CONCLUSIONS

The proposed dry friction models enables as well to describe the relationship between force and kinematical characteristics by smooth analytical functions over the entire range of angular and linear velocities as to take into account the more realistic representation about normal contact stresses distribution and differential characteristics of the friction law. The approximate models preserve all analytical properties of the models based on the exact integral expressions and correctly describe the behaviour of the net vector and torque of the friction forces and their first derivatives at zero and infinity.

Moreover, the models coefficients are numbers that can be identified from experiments. Consequently, these models may be considered as phenomenological models of the combined dry friction.

The procedure of approximate dry friction models construction is universal method of the dry friction model developing in more difficult cases.

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